Section #19 Linear versus Exponential

Goals

Anyone thinking that business is a route to avoid numbers and math is greatly mistaken. Businesses analyze graphs and table of data on a daily basis. The ability to recognize patterns in sets of numbers is a valuable skill to have when making business projections and decisions. A goal of this section is to introduce a method for distinguishing linear and exponential sets of data by analyzing the change in a sequence of data. Also, the formula for exponential relationships is introduced.

Key Questions

Suppose you want to select a company in which to invest. One way to choose a company is to analyze the profit of the company over a number of periods. Based on the following data in Table 19.1, which company will have the greatest profit in the 6th period (thus, possibly a better return for your investment)? Additionally, which company will have the greatest profit in the 12th period (in case it is a long-term investment)?

Question 19.1: Without doing any involved calculations, rank (in the order of most to least) the four companies in terms of your projection of their profits for the 6th period.

Ranking: _________ _________ _________ _________

Table 19.1: Profit ($ millions) per period

Company	**Period**						
	1	**2**	**3**	**4**	**5**	**6**	**12**
A	20	24	28	32	36		
B	2.25	4.5	9	18	36		
C	4/9	4/3	4	12	36		
D	8	15	22	29	36		

Question 19.2: In terms that your peers in the class can understand, explain your reasoning for your ranking.

Let's begin to conduct a more systematic approach for analyzing the given data. For this section, we'll look at two types of patterns to sequences of data. There are many more types of sequences, but to simplify things we will examine only two.

Pattern Name	Description
Additive (Arithmetic Sequence)	There is a number r such that each term is obtained by adding r to the term before it. The number r is called the "*increment*" of the sequence.
Multiplicative (Geometric Sequence)	There is a number m such that each term is obtained by multiplying m to the term before it. The number m is called the "*multiplier*" of the sequence.

Question 19.3: For the following ***additive*** sequences in Table 19.2, determine the increment and the 6^{th} term.

Table 19.2

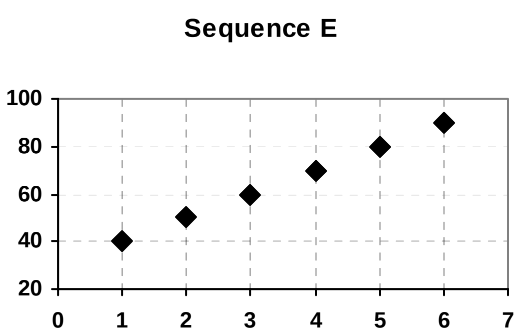
Sequence	**Term**						**Increment value**
	1	**2**	**3**	**4**	**5**	**6**	
E	40	50	60	70	80		
F	64	71	78	85	92		

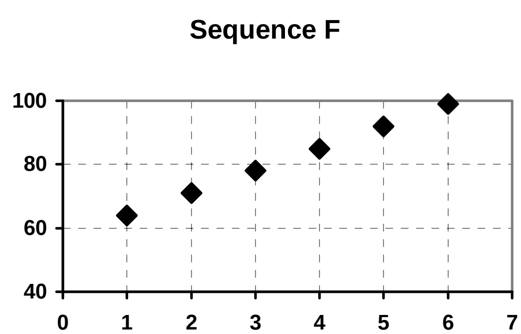
And, now it's time to introduce some notation. When you found the value of the increment for the sequences, you first took the difference of two consecutive terms. Then to get the 6^{th} term, you added the increment to the 5^{th} term. Like most things in mathematics, there is notation that represents these two steps. Let's call the E-sequence, E(k). The letters i, j, and k are normally used to denote the terms in the sequence; $E(3)$ is the third term in the E-sequence. So, the difference between consecutive terms in a sequence is denoted by $E(k)-E(k-1)$. For the E-sequence, the increment can be found using the 2^{nd} and 3^{rd} terms: $E(3)-E(2)=60-50=10$. The 6^{th} term would be represented and calculated by $E(6)=E(5)+10=80+10=90$. (Translation: the 6^{th} term of the E-sequence is calculated by adding 10 to the 5^{th} term.) The following notation is used to indicate the method for finding any term in the sequence: $E(k)=E(k-1)+10$. This formula is called a ***Recursive*** formula because the calculation for a term uses the value of the previous term. Note: Another notation for sequence is to use subscripts; E_6 represents the 6th term in the E-sequence.

Question 19.4: To see if you understand the labeling system, write the notation for determining the increment of the F-sequence when evaluating the difference of a term and the next term.

Question 19.5: To see if you can express the relationships and can apply the terms, consider this scenario: Suppose you are working with another student in the class. The other student picked the 4th and 5th terms to calculate the increment of the E-sequence. Does it matter that you picked two other terms to calculate the increment value? In your explanation, incorporate the terms *additive pattern* and *increment*.

Now let's look at another form for expressing this relationship. One of the things that a person wants to do when confronted with a table of data is to make a mathematical model of the data that can be used to make predictions. (*To think, people say a model wouldn't date a mathematician.*) So, what kind of mathematical relationship is an additive sequence? Plotting the sequence with the term number (or, k-value) on the horizontal axis and the term value on the vertical axis will make the linear relationship more apparent. The following are the graphs of the two sequences, E and F.





Question 19.6: Determine the equation of the line for both sequences, E(k) and F(k). (Answer: $E(k) = 10(k-3) + 60 = 10k + 30$)

Let's look at some patterns between the sequences in the tables and the equations. First, notice the connection between the increment of the E-sequence and the slope of the line. Yes, they're the same thing. Second, the 30 from the equation is not only the y-intercept of the graph but also the initial value of the E-sequence. Recall that the y-intercept (in this case) is the point (0, 30). So, term 0 in the E-sequence has the value of 30 and is called the ***initial value***. The use of initial will become more apparent when time is introduced.

The equations that we just developed are called ***Explicit*** formulas, because you can input the number of the term to obtain the term value. You don't have to know the previous term to get the one that you want. (*And to think, some rock band sang that "you can't always get what you want."*) If you wanted to find the 10^{th} term, just substitute 10 for k in the formula. You don't have to go through and calculate all of the terms in between to get to the 10^{th} term.

Question 19.7: Calculate the 12^{th} terms for the E and F sequences.

Question 19.8: Let's return to the data in Table 19.1. Determine which two of the sequences in the table are additive sequences. Then, develop the explicit formulas for each of the two sequences. Finally, calculate the projected profit for the 12^{th} period. (That's assuming the pattern remains the same throughout the periods.)

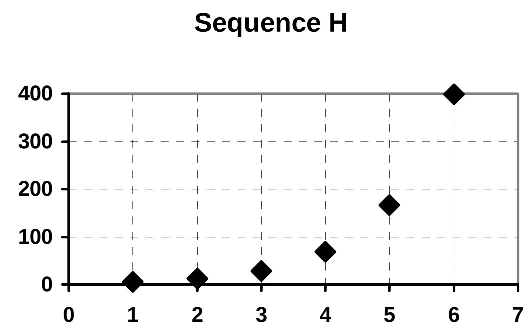
Now that we've analyzed the additive sequences, it's on to the ***multiplicative sequences***. Hopefully, you will discover that there are some similarities between the two types of sequences.

Question 19.9: Recall, when working with additive sequences, you subtracted two consecutive terms to determine the increment value. Intuitively, what will be the approach for finding the multiplier within a multiplicative sequence?

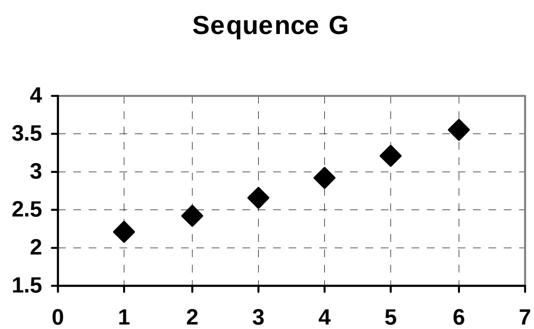
Question 19.10: For the following multiplicative sequences in Table 19.3, determine the multiplier and the 6^{th} term.

Table 19.3

Sequence	**Term**						**Multiplier value**
	1	**2**	**3**	**4**	**5**	**6**	
G	2.2	2.42	2.662	2.9282	3.22102		
H	5	12	28.8	69.12	165.888		



In terms of the notation, dividing two consecutive terms to find the multiplier is denoted by $m = \dfrac{G(k)}{G(k-1)}$, which is 1.1 for the G-sequence. Like the situation with the additive sequences, it does not matter which two consecutive terms we pick; the value is the same. And, the recursive formula for calculating the value of any term is given by $G(k) = G(k-1) \times m$. The calculation for determining the 6$^{\text{th}}$ term is given by $G(6) = G(5) \times m = 3..22102 \times 1.1 = 3.543122$.



Recall, we discovered that additive sequences were really linear relationships, just in a different format. Well, multiplicative sequences also have an alternative identity: multiplicative sequences are exponential relationships. The graphs of the G and H sequences appear to be exponential.

The graph of the H-sequence looks like a typical exponential relationship, but the graph of the G-sequence almost looks linear. So, how does a person tell if it's linear or exponential from a graph? One method is to analyze the slopes of secants connecting two consecutive points.

Question 19.11: Draw the secant line connecting points at k = 1 and k = 2. Then, draw the secant line connecting points at k = 2 and k = 3. Repeat process. With regards to the slopes of the secants that you drew, what is the pattern?

Question 19.12: This pattern is also evident numerically. In the Table 19.4, determine the difference in the values of consecutive terms. (Recall, ΔG denotes change in the values of the G-sequence.) Does the differences follow the same pattern as the slopes of the secants?

Table 19.4

Sequence	**Term**				
	1	**2**	**3**	**4**	**5**
G	2.2	2.42	2.662	2.9282	3.22102
ΔG					

Note: the graphical analysis that we just performed does not verify that the sequence was exponential. Other types of functions also have secants whose slopes increase. The graphical analysis of the G and H sequences provided initial evidence that they were not linear and could be exponential. The main idea indicating that the G and H sequences exponential was that they had a multiplicative pattern.

Now, it's time for the introduction of the symbolic form for exponential relationships. The standard form for an exponential relationship is $S(k) = S_0 \times m^k$, where S_0 is the initial value, m is the multiplier (also called the base of the exponential), and k is the term of the sequence. To make the explicit formula for the G-sequence, we substitute the value of the multiplier: $G(k) = G_0 \times 1.1^k$. The next thing to do is to determine the initial value. In some problem situations, the initial value is given. But, in this example, we'll have to substitute a point into the formula and solve for the initial value. Let's use the point (1, 2.2).

$$G(k) = G_0 \times 1.1^k \;\rightarrow\; 2.2 = G_0 \times 1.1^{(1)} \;\rightarrow\; \frac{2.2}{1.1} = G_0 \;\rightarrow\; G_0 = 2$$

So, the explicit formula for the G-sequence is $G(k) = 2 \times 1.1^k$.

Question 19.13: Develop the explicit formula for the H-sequence given in Table 19.3.

Question 19.14: Let's return to the data in Table 19.1, concerning companies' profit. For the remaining two sequences, confirm that they are multiplicative sequences. Then, develop the explicit formulas for each of the sequences. Finally, calculate the projected profit for the 12^{th} period. (That's assuming the pattern remains the same throughout the periods.)

Question 19.15: In a paragraph, compare and contrast the explicit formulas for an additive sequence and for a multiplicative sequence. Explain what makes them similar and what makes them different? Incorporate the terms *initial value*, *increment*, *slope*, *base*, and *multiplier*.

Within this section, you have developed the skills to distinguish whether a set of data in a table is linear or exponential. Once you've made the distinction, you have a procedure for developing the explicit formula that represents the data and can be used to make predictions. Now, apply these skills and concepts to answer the following economic situation. The question may be challenging because it asks you to synthesize the concepts encountered in this section.

Question 19.16: In 1798, Thomas Robert Malthus wrote *Essay on Population*. "Famine seems to be the last, the most dreadful resource of nature. The power of population is so superior to the power of the earth to provide subsistence…that premature death must in some shape or other visit the human race." Reverend Malthus had a gloomy outlook because he considered the human population to grow exponentially. Whereas, the food supply would grow only linearly due to limitations for cultivating land. Using the concepts covered in this section and behavioral patterns of the data in Table 19.1, decide whether Reverend Malthus was a major pessimistic or was justified in his concerns.